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Lagrange characteristic method for solving a class of nonlinear partial differential equations of fractional order

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Abstract

We propose an extension of the Lagrange method of characteristics for solving a class of nonlinear partial differential equations of fractional order. This refers to the Lagrange method of the auxiliary system for linear fractional partial differential equations (which is given in an appendix). The key to the approach is the Taylor's series of fractional order $f(x+h) = E_\alpha(h^\alpha D_x^\alpha)f(x)$ where E_α is the Mittag-Leffler function.

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1. Introduction

Until now, two methods have been more extensively used for solving fractional partial differential equations (FPDE): Laplace and Fourier transformation on the one hand, and the separation of variables on the other hand [1–3]. Let us mention also solutions in the form of series of functions [11]. Of course, each of these methods has its own advantages and its own defects, and as a result, alternatives may be useful in some instances, for getting different insights into the problem. On this basis, we show here how the so-called Lagrange (Charpit) method of characteristics, which is used for solving nonlinear PDE of first order, can be extended to fractional PDE. This result could serve as a point of departure for expanding the variational theory of fractional optimal control.

This short note is organized as follows. For the convenience of the reader, in the next section, we shall give a summary of the fractional Taylor's series and its consequences, and then we shall describe the method itself. The latter refers to Lagrange's technique of the auxiliary system for linear FPDE, and this method is described in [Appendix A](#).

In the following, we shall use at will and for convenience, depending upon the context, the notation $\partial^\alpha f(x, y)/\partial x^\alpha$, $\partial_x^\alpha f(x, y)$ and $f^{(\alpha)}(x, y)$, $0 < \alpha \leq 1$.

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2. Background on Taylor's series of fractional order

2.1. Fractional derivatives

Definition 2.1. Let $f : \Re \rightarrow \Re$, $x \rightarrow f(x)$, denote a continuous function. Its fractional derivative of order α is defined by the following expression [6–9] (the symbol $:=$ means that the left side is defined by the right one):

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} f(\xi) d\xi, \quad \alpha < 0. \quad (2.1)$$

For positive α , we set

$$f^{(\alpha)}(x) = \left(f^{(\alpha-1)}(x) \right)', \quad 0 < \alpha < 1, \quad (2.2)$$

$$f^{(\alpha)}(x) := \left(f^{(\alpha-n)}(x) \right)^{(n)}, \quad n < \alpha < n+1, n \geq 2. \quad \square \quad (2.3)$$

Definition 2.2. Consider the function of Definition 2.1; and let $h > 0$ denote a constant discretization span. Define the forward operator $FW(h)$,

$$FW(h).f(x) := f(x+h);$$

then the fractional difference of order α , $\alpha \in \Re$, $\alpha > 0$, of $f(x)$ is defined by the expression

$$\begin{aligned} \Delta^\alpha.f(x) &:= (FW - 1)^\alpha.f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h]. \quad \square \end{aligned} \quad (2.4)$$

Lemma 2.1. The following equality holds:

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha}. \quad \square \quad (2.5)$$

The proof can be obtained by using the Laplace transform and Z-transform and then making h tend to zero. See for instance [3,4].

2.2. Taylor expansion of fractional order

A generalized Taylor expansion of fractional order reads as follows:

Proposition 2.1. Assume that the continuous function $f : \Re \rightarrow \Re$, $x \rightarrow f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any α , $0 < \alpha < 1$; then the following equality holds:

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1 + \alpha k)} f^{(\alpha k)}(x), \quad 0 < \alpha < 1. \quad (2.6)$$

where $f^{(\alpha k)}$ is the derivative of order αk of $f(x)$. \square

Formally, one has

$$f(x+h) = E_\alpha(h^\alpha D_x^\alpha) f(x). \quad (2.7)$$

where $E_\alpha(x)$ denotes the Mittag-Leffler function defined by the expression

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + \alpha k)}. \quad (2.8)$$

Lemma 2.1. Assume that $m < \alpha < m + 1$, $m \in N - \{0, 1\}$; then

$$f^{(m)}(x+h) = \sum_{k=0}^{\infty} \frac{h^{k(\alpha-m)}}{\Gamma[1+k(\alpha-m)]} D^{k(\alpha-m)} f^{(m)}(x), \quad m < \alpha < m+1. \quad \square \quad (2.9)$$

Making $m = 1$ into Eq. (2.9) and integrating the results so obtained with respect to h yields

$$f(x+h) = f(x) + hf'(x) + \sum_{k=1}^{\infty} \frac{h^{k\beta+1}}{\Gamma(k\beta+2)} f^{(k\beta+1)}(x), \quad \beta := \alpha - 1. \quad (2.10)$$

As a direct application, one has the following:

Lemma 2.2. Assume that $f(x)$, in Proposition 2.1, is α th-differentiable; then the following equalities hold:

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha} = \Gamma(1+\alpha) \lim_{h \downarrow 0} \frac{\Delta f(x)}{h^\alpha}, \quad 0 < \alpha < 1. \quad (2.11)$$

and

$$f^{(\alpha)}(x) = \Gamma[1+(\alpha-m)] \lim_{h \downarrow 0} \frac{\Delta f^{(m)}(x)}{h^{\alpha-m}}, \quad m < \alpha < m+1. \quad \square \quad (2.12)$$

For the proof of these results, see [5,6].

Useful relations

Eq. (2.6) provides the useful relation

$$\Delta^\alpha f \cong \Gamma(1+\alpha) \Delta f, \quad 0 < \alpha < 1, \quad (2.13)$$

or in a like manner $d^\alpha f \cong \Gamma(1+\alpha) df$, between the fractional difference and finite difference.

Assume now that $1 < \alpha \leq 2$. Then Eq. (2.10) yields

$$\Delta f \cong f'(x)\Delta x + \frac{(\Delta x)^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(x), \quad (2.14)$$

where f' denotes the derivative of f ; therefore

$$\Delta^\alpha f \cong \Gamma(1+\alpha)(\Delta f - f'(x)\Delta x). \quad (2.15)$$

The following formulas will be useful later:

$$D^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha}, \quad (2.16)$$

$$(x(t)y(t))^{(\alpha)} = x^{(\alpha)}(t)y(t) + x(t)y^{(\alpha)}(t), \quad (2.17)$$

$$(f[x(t)])^{(\alpha)} = f'_x(x)x^{(\alpha)}(t), \quad (2.18)$$

which are direct consequences of the equality $d^\alpha x(t) = \Gamma(1+\alpha)dx(t)$.

Remarks. (i) Osler [10] has previously proposed a generalization of Taylor's series in the complex plane, in the form

$$f(z) = \alpha \sum_{k=-\infty}^{k=+\infty} \frac{f^{(\alpha k)}(z_0)}{\Gamma(1+\alpha k)} (z-z_0)^{\alpha k}, \quad (2.19)$$

which, in the *useful* case when $f(z)$ is continuous at z_0 , reduces to

$$f(z) = \alpha \sum_{k=0}^{k=+\infty} \frac{f^{(\alpha k)}(z_0)}{\Gamma(1+\alpha k)} (z-z_0)^{\alpha k}. \quad (2.20)$$

So, in the special case where $z = z_0$, this Eq. (2.20) would provide

$$f(z_0) = \alpha f(z_0)?$$

In addition, we shall point out that, whilst Osler uses contours of integration in the complex plane to obtain his series, here, by means of a simple symbolic calculus, we derive a formula in a very compact form, in terms of the Mittag–Leffler function, namely Eq. (2.7).

2.3. Integration with respect to $(dt)^\alpha$

The solution of the equation

$$dx = f(t)(dt)^\alpha, \quad t \geq 0, \quad x(0) = x_0, \quad (2.21)$$

is defined by the following result

Lemma 2.2. *Let $f(t)$ denote a continuous function; then the solution of Eq. (2.21) is defined by the equality*

$$\int_0^t f(\tau)(d\tau)^\alpha = \alpha \int_0^t (t - \tau)^{\alpha-1} f(\tau)d\tau, \quad 0 < \alpha \leq 1. \quad \square \quad (2.22)$$

Some examples

On making $f(\tau) = 1$ in (2.22) one obtains

$$\int_0^t (d\tau)^\alpha = t^\alpha, \quad 0 < \alpha \leq 1. \quad (2.23)$$

Assume now that $f(t)$ is the Dirac delta generalized function $\delta(t)$; then one has

$$\int_0^t \delta(\tau)(d\tau)^\alpha = \alpha t^{\alpha-1}, \quad 0 < \alpha \leq 1. \quad (2.24)$$

3. Lagrange method for nonlinear FPDE

Statement of the problem

Our purpose is to obtain the solution of the fractional partial differential equation

$$F(x, y, u, p, q) = 0, \quad (3.1)$$

where F is a real-valued function, $x \in \Re$ and $y \in \Re$ are two independent variables, $u(x, t)$ is the unknown function to be determined, p is the fractional partial derivative of $u(x, y)$ w.r.t. x ,

$$p = \frac{\partial^\alpha u}{\partial x^\alpha} \equiv \partial_x^\alpha u \equiv u_x^{(\alpha)}, \quad 0 < \alpha \leq 1$$

and q is the partial derivative of u w.r.t. y ; $q = \partial_y u = u_y$.

Lemma 3.1 (Main Result: Auxiliary System for Solution). *Solving (3.1) amounts to solving the following system of equations:*

$$\frac{(dx)^\alpha}{F_p} = \frac{d^\alpha y}{F_q} = \frac{d^\alpha u}{pF_p + qF_q} = -\frac{d^\alpha p}{F_x^{(\alpha)} + pF_u} = -\frac{d^\alpha q}{F_y + qF_u}. \quad \square \quad (3.2)$$

Derivation of the Eq. (3.2)

(i) The idea is to look for another function G such that

$$G(x, y, u, p, q) = C, \quad (3.3)$$

where C denotes a constant. Strictly speaking, F and G define $p = \varphi(x, y, u)$ and $q = \psi(x, y, u)$ as functions of x, y and u , and then their integration will provide u . In order to determine G we shall write that the following consistency condition is satisfied:

$$\partial_y (\partial_x^\alpha u) = p_y + p_u q = \partial_x^\alpha (\partial_y u) = q_x^{(\alpha)} + q_u p, \quad (3.4)$$

and to this end we need the explicit expressions for the partial derivatives p and q , which will be obtained by equating to zero the partial differentials $d_x F, d_y F, d_u F$ and similarly for G .

(ii) $d_y F$ and $d_u F$ are obtained by the usual chain rule; for instance

$$d_y F = (F_y + F_p p_y + F_q q_y) dy.$$

As regards $d_x F$, one has (on combining fractional and nonfractional Taylor series)

$$\begin{aligned} d_x F &= \Gamma^{-1}(1 + \alpha) F_x^{(\alpha)} (dx)^\alpha + F_p d_x p + F_q d_x q \\ &= \Gamma^{-1}(1 + \alpha) \left(F_x^{(\alpha)} (dx)^\alpha + F_p d_x^\alpha p + F_q d_x^\alpha q \right) \\ &= \Gamma^{-1}(1 + \alpha) \left(F_x^{(\alpha)} + F_p p_x^{(\alpha)} + F_q q_x^{(\alpha)} \right) (dx)^\alpha. \end{aligned}$$

We then have the following set of equations:

$$F_x^{(\alpha)} + F_p p_x^{(\alpha)} + F_q q_x^{(\alpha)} = 0, \quad (3.5)$$

$$F_y + F_p p_y + F_q q_y = 0, \quad (3.6)$$

$$F_u + F_p p_u + F_q q_u = 0, \quad (3.7)$$

and

$$G_x^{(\alpha)} + G_p p_x^{(\alpha)} + G_q q_x^{(\alpha)} = 0, \quad (3.8)$$

$$G_y + G_p p_y + G_q q_y = 0, \quad (3.9)$$

$$G_u + G_p p_u + G_q q_u = 0. \quad (3.10)$$

These Eqs. (3.5)–(3.10) provide

$$(F_p G_q - F_q G_p) p_y = F_q G_y - F_y G_q, \quad (3.11)$$

$$(F_p G_q - F_q G_p) q_x^{(\alpha)} = F_x^{(\alpha)} G_p - F_p G_x^{(\alpha)}, \quad (3.12)$$

$$(F_p G_q - F_q G_p) p_u = F_q G_u - F_u G_p, \quad (3.13)$$

$$(F_p G_q - F_q G_p) q_u = F_u G_p - F_p G_u. \quad (3.14)$$

and on substituting into (3.4) we have the fractional PDE

$$F_p G_x^{(\alpha)} + F_q G_y + (p F_p + q F_q) G_u - \left(F_x^{(\alpha)} + p F_u \right) G_p - (F_y + q F_u) G_q = 0. \quad (3.15)$$

The auxiliary system associated with this equation (see [Appendix A](#)) is exactly (3.20). \square

Example

Let us consider the equation

$$p^2 + q^2 - (x + y) = 0. \quad (3.16)$$

Its solution can be obtained directly by rewriting (3.16) in the form

$$p^2 - x = y - q^2 = \beta$$

and therefore

$$du = \Gamma^{-1}(1 + \alpha) \sqrt{x + \beta} (dx)^\alpha + \sqrt{y - \beta} dy. \quad (3.17)$$

Let us use now Lagrange's auxiliary system (3.2). We then have

$$\frac{(dx)^\alpha}{2p} = \frac{d^\alpha y}{2q} = \frac{d^\alpha u}{2(p^2 + q^2)} = -\frac{d^\alpha p}{-d^\alpha x/dx^\alpha} = -\frac{d^\alpha q}{-1}. \quad (3.18)$$

Therefore we obtain the two equations

$$2p (d^\alpha p) = d^\alpha x, \quad (3.19)$$

$$2q (d^\alpha q) = d^\alpha y. \quad (3.20)$$

Integrating (3.19) yields

$$\int 2p p_x^{(\alpha)} (dx)^\alpha = \int x^{(\alpha)} (dx)^\alpha$$

and therefore

$$p = x + \text{const}. \quad (3.21)$$

In like manner, one has

$$q = y + \text{const}, \quad (3.22)$$

and on substituting this into (3.16), we still obtain (3.17).

4. Concluding remarks

As a concluding remark we shall point out that we now have at hand all the framework for expanding a variational approach to fractional optimal control via the fractional Hamilton–Jacobi equation

$$G_t^{(\alpha)} + \sum_{j=1}^n f_j(x, u, t) G'_{x_j} + g(x, u, t) = 0. \quad (4.1)$$

and mainly in this way one should be able to develop a theory of Lagrangian mechanics for non-random fractional dynamics, on which we are now working.

Appendix A. Lagrange method for fractional linear PDE

A.1. Statement of the problem

Our purpose in the following is to determine the solution of the FPDE

$$f(x, y, u) \frac{\partial^\alpha u}{\partial x^\alpha} + g(x, y, u) \frac{\partial u}{\partial y} = h(x, y, u), \quad 0 < \alpha < 1, \quad (A.1)$$

where $u : \mathfrak{R}^2 \rightarrow \mathfrak{R}$, $(x, y) \rightarrow u(x, y)$, is subject to the initial condition

$$u(x, 0) = u_0(x). \quad (A.2)$$

In the special case where $\alpha = 1$, the most useful approach for solving for $u(x, y)$ is the Lagrange technique which introduces the auxiliary system

$$\frac{dx}{f} = \frac{dy}{g} = \frac{du}{h}, \quad (A.3)$$

and looking for two integral functions $\Phi_1(x, y, u) = \text{const}$ and $\Phi_2(x, y, u) = \text{const}$ which provide the general solution in the form $F(\Phi_1, \Phi_2)$. In the following, we shall show how this method can be modified to solve Eq. (A.1). To this end, we need a preliminary result on systems of fractional (ordinary) differential equations.

A.2. Systems of fractional differential equations

Let us consider the system

$$\dot{x}(t) = f_1(x, y, t), \quad x(0) = x_0, \quad (\text{A.4})$$

$$y^{(\alpha)}(t) = f_2(x, y, t), \quad y(0) = y_0, \quad 0 < \alpha < 1, \quad (\text{A.5})$$

where t denotes time, to fix thought. We have the following result:

Lemma A.1. Assume that $\Psi(x, y, t) = \text{const}$ is a first integral for the system (A.4) and (A.5); then one has the following FPDE:

$$f_1^\alpha \frac{\partial^\alpha \Psi}{\partial x^\alpha} + f_2 \frac{\partial \Psi}{\partial y} + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \frac{\partial \Psi}{\partial t} = 0. \quad \square \quad (\text{A.6})$$

Proof. Applying the operator $E_\alpha(dx^\alpha D_x^\alpha) \exp(dy D_y) \exp(dt D_t)$ to the function Ψ , we get the increment

$$\begin{aligned} d\Psi &= \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \Psi}{\partial x^\alpha} (dx)^\alpha + \frac{\partial \Psi}{\partial y} dy + \frac{\partial \Psi}{\partial t} dt \\ d\Psi &= \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \Psi}{\partial x^\alpha} (dx)^\alpha + \frac{1}{\Gamma(1+\alpha)} \frac{\partial \Psi}{\partial y} d^\alpha y + \frac{1}{\Gamma(1+\alpha)} \frac{\partial \Psi}{\partial t} d^\alpha t \\ d\Psi &= \frac{1}{\Gamma(1+\alpha)} \left(\frac{\partial^\alpha \Psi}{\partial x^\alpha} \left(\frac{dx}{dt} \right)^\alpha + \frac{\partial \Psi}{\partial y} \frac{d^\alpha y}{dt^\alpha} + \frac{\partial \Psi}{\partial t} \frac{d^\alpha t}{dt^\alpha} \right) (dt)^\alpha, \end{aligned} \quad (\text{A.7})$$

and we remark that

$$\frac{d^\alpha t}{dt^\alpha} = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad (\text{A.8})$$

for obtaining Eq. (A.6). \square

A.3. Auxiliary system associated with FPDE

We now come back to the fractional PDE (A.1), and we denote by $\Phi(x, y, u)$ the first integral (function), namely $\Phi(x, y, u) = \text{constant}$. We have the following result:

Lemma A.2. Let $\Phi(x, y, u) = K$ denote a first integral function for the FPDE (4.1); then it satisfies the FPDE

$$f \frac{\partial^\alpha \Phi}{\partial x^\alpha} + g \frac{\partial \Phi}{\partial y} + h \frac{\partial \Phi}{\partial u} = 0. \quad \square \quad (\text{A.9})$$

Proof. (i) Let $d_x \Phi$ denote the increment of Φ along x only. We then have

$$\begin{aligned} d_x \Phi &= \frac{\partial \Phi}{\partial u} d_x u + \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \Phi}{\partial x^\alpha} (dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{\partial \Phi}{\partial u} d_x^\alpha u + \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \Phi}{\partial x^\alpha} (dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \left(\frac{\partial \Phi}{\partial u} \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha \Phi}{\partial x^\alpha} \right), \end{aligned}$$

and on making explicit the condition $d_x \Phi = 0$, we obtain

$$\frac{\partial^\alpha u}{\partial x^\alpha} = - \frac{\Phi_x^{(\alpha)}}{\Phi'_u}. \quad (\text{A.10})$$

(ii) The same (standard) calculation w.r.t. y yields

$$d_y \Phi = \left(\frac{\partial \Phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \Phi}{\partial y} \right) dy,$$

and therefore, on equating to zero,

$$\frac{\partial u}{\partial y} = -\frac{\Phi'_y}{\Phi'_u}. \quad (\text{A.11})$$

(iii) Substituting (A.10) and (A.11) into (A.1) yields the result (A.9). \square

Auxiliary system associated with the FPDE

Lemma A.3. *The auxiliary system of partial differential equations associated with the FPDE (A.1) is*

$$\frac{(dx)^\alpha}{f} = \frac{d^\alpha y}{g} = \frac{(du)^\alpha}{h}. \quad \square \quad (\text{A.12})$$

Derivation of these equations

The key idea is provided by the similarity between the FPDE (A.6) and (A.9). More explicitly, we rewrite (A.9) in the form

$$\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \left(\frac{f^\alpha}{h} \frac{\partial^\alpha \Phi}{\partial x^\alpha} + \frac{g}{h} \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial u} \right) = 0. \quad (\text{A.13})$$

On comparing (A.13) with (A.6), we are led to make the substitution $t \leftarrow u$ and to set

$$f_1^\alpha(x, y, u) \equiv \frac{u^{1-\alpha} f(x, y, u)}{\Gamma(2-\alpha)h(x, y, u)},$$

$$f_2(x, y, u) \equiv \frac{u^{1-\alpha} g(x, y, u)}{\Gamma(2-\alpha)h(x, y, u)}.$$

We then have the associated differential equations (the parallels of (A.4) and (A.5))

$$\left(\frac{dx}{du} \right)^\alpha = \frac{u^{1-\alpha} f(x, y, u)}{\Gamma(2-\alpha)h(x, y, u)}, \quad (\text{A.14})$$

$$\frac{d^\alpha y}{du^\alpha} = \frac{u^{1-\alpha} g(x, y, u)}{\Gamma(2-\alpha)h(x, y, u)}, \quad (\text{A.15})$$

and therefore the associated system (A.12).

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